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OPTIMAL STRUCTURAL DESIGN FOR GIVEN DYNAMIC DEFLECTION*

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Abstract—This paper is concerned with optimal design of structures of various types that are excited to harmonic vibrations by a single load, the intensity of which varies harmonically with time. Amplitude and frequency of this load are prescribed as well as the "dynamic response" of the structure, which is defined as the virtual work of the load amplitude on the displacement amplitude of its point of application. Subject to this "design constraint" the structure is to use the smallest possible amount of a given structural material. Necessary and sufficient optimality conditions are given for rods and beams with continuously varying or segmentwise constant cross sections, and for trusses where all masses are lumped at the joints. Examples are presented, which illustrate the possible saving in structural weight.

1. INTRODUCTION

THE general problem of minimum-weight design of a structure involves specifications of (1) the purpose (or purposes) of the structure, (2) the design constraint (or constraints), (3) the type of the desired structure, and, possibly, (4) its general shape. Typical examples in these categories are: (1) transmission of given loads to given foundations, or support of given masses; (2) upper bounds for deflections of specified points or for maximum deflection, or lower bounds for buckling load or fundamental natural frequency; (3) rod, truss, beam, arch, plate, or shell; (4) shape of centerline of arch or median surface of shell.

The present paper deals with single-purpose structures that have to transmit a load of harmonically varying intensity to given points of support. The design constraint sets an upper bound on the "dynamic response" which is here defined as the virtual work of the given amplitude of the load on the amplitude of the deflection of its point of application. Minimum-weight design of rods, beams, and trusses is discussed.

There is a substantial body of literature on optimal structural design; but this particular design constraint, though of considerable practical importance, does not seem to have been treated. The related design constraint, which imposes an upper bound on the static deflection under a given load, has been discussed by Barnett [1], and Haug and Kirmser [2]. Another related constraint, which sets a lower bound on the fundamental natural frequency, has been treated by Niordson [3], Turner [4], Taylor [5], Zarghamee [6], and Sheu [7].

The conventional approach to problems of optimal structural design is through the classical calculus of variations, which readily furnishes a necessary optimality condition. The arguments establishing the sufficiency of this condition, if it is indeed sufficient, may be rather involved. In the present paper a new method of establishing optimality criteria is

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used that has been developed in recent papers by Prager and Taylor [8], and Prager and Shield [9].

This method is applicable whenever there exists an extremum principle characterizing the quantity whose value is prescribed or bounded by design constraints. This procedure consists of two parts: the integration of a nonlinear differential equation for the displacement field of the optimal structure, and the determination of the optimal specific stiffness from the linear differential equation of motion and the already obtained displacement field.

Sections 2 and 3 deal with optimal design of rods for given dynamic response. Rods of continuously varying or segmentwise constant cross section are discussed. Section 4 is concerned with optimal design of beams for given dynamic response. Here only beams of continuously varying cross section are considered. Section 5, finally treats optimal design of a truss for given dynamic response when the layout of the bars of the truss is given and all masses are lumped at the joints.

2. ROD OF CONTINUOUSLY VARYING CROSS SECTION

Figure 1 shows a rod with continuously varying cross section that is fixed at one end ($x = 0$) and subjected to the axial load $P \cos \omega t$ at the other end ($x = l$). Both the amplitude

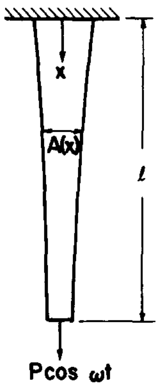


FIG. 1. Rod of continuously varying cross section.

P and the frequency ω are given, and *ω is to be smaller than the fundamental natural frequency ω_1* of the rod. Writing the axial displacement as $u(x) \cos \omega t$, we define the dynamic response of the rod to the given load as $Pu(l)$. As has been mentioned, the method of Prager and Taylor [8] presupposes that the design constraint, that is the required value of $Pu(l)$, be characterized by an extremum principle. Such a principle will now be established.

The differential equation of motion is [10]

$$[EA(x)u'(x)]' + \omega^2 \rho A(x)u(x) = 0, \tag{2.1}$$

where E is Young's modulus, ρ is the density, and a prime denotes differentiation with respect to x. The boundary conditions are

$$\begin{aligned} &\text{at} \quad x = 0: \qquad u(0) = 0, \\ &\text{at} \quad x = l: \qquad EA(l)u'(l) = P. \end{aligned} \tag{2.2}$$

According to Rayleigh's principle, the fundamental natural frequency ω_1 of the rod is given by

$$\omega_1^2 = \min\left\{\frac{\int_0^l EA(x)\bar{u}'^2(x)\,dx}{\int_0^l \rho A(x)\bar{u}^2(x)\,dx}\right\} \tag{2.3}$$

over all kinematically admissible $\bar{u}(x)$. A continuous function $\bar{u}(x)$ in this problem is called kinematically admissible if it is twice differentiable with respect to x and satisfies the boundary condition $\bar{u}(0) = 0$; it need not, however, fulfill any boundary condition at $x = l$.

Since ω_1 is assumed to exceed ω, it follows from (2.3) that the functional

$$F[\bar{u}(x)] = \int_0^l EA(x)\bar{u}'^2(x)\,dx - \omega^2 \int_0^l \rho A(x)\bar{u}^2(x)\,dx \tag{2.4}$$

is positive definite. For the actual displacement field $u(x)$ and any kinematically admissible displacement field $\bar{u}(x)$, we therefore have

$$\int_0^l EA(x)[\bar{u}'(x)-u'(x)]^2\,dx - \omega^2 \int_0^l \rho A(x)[\bar{u}(x)-u(x)]^2\,dx \geq 0. \tag{2.5}$$

Expansion of the squared terms and subtraction of $2\int_0^l EA(x)u'^2(x)\,dx - 2\omega^2\int_0^l \rho A(x)u^2(x)\,dx$ from both sides of the resulting inequality yields

$$\begin{aligned}\int_0^l EA\bar{u}'^2\,dx - \omega^2\int_0^l \rho A\bar{u}^2\,dx - 2\int_0^l EAu'\bar{u}'\,dx + 2\omega^2\int_0^l \rho Au\bar{u}\,dx \\ \geq \int_0^l EAu'^2\,dx - \omega^2\int_0^l \rho Au^2\,dx - 2\int_0^l EAu'^2\,dx + 2\omega^2\int_0^l \rho Au^2\,dx.\end{aligned} \tag{2.6}$$

Using integration by parts in addition to the differential equation of motion (2.1) and the boundary conditions (2.2), we finally obtain the following minimum principle:

$$\begin{aligned}\int_0^l EA(x)\bar{u}'^2(x)\,dx - \omega^2\int_0^l \rho A(x)\bar{u}^2(x)\,dx - 2P\bar{u}(l) \\ \geq \int_0^l EA(x)u'^2(x)\,dx - \omega^2\int_0^l \rho A(x)u^2(x)\,dx - 2Pu(l) = -Pu(l).\end{aligned} \tag{2.7}$$

Note that this minimum principle reduces to the well-known principle of minimum potential energy if $\omega = 0$, which corresponds to static loading. Note also that the dynamic response $Pu(l)$ is the minimum of the functional $F[\bar{u}(x)]$.

Now consider two designs $A(x)$ and $\hat{A}(x)$ satisfying the design constraint of having the same dynamic response. Accordingly

$$\begin{aligned}\int_0^l EA(x)u'^2(x)\,dx - \omega^2\int_0^l \rho A(x)u^2(x)\,dx \\ = \int_0^l E\hat{A}(x)\hat{u}'^2(x)\,dx - \omega^2\int_0^l \rho\hat{A}(x)\hat{u}^2(x)\,dx,\end{aligned} \tag{2.8}$$

where $u(x)$ and $\hat{u}(x)$ are the displacement fields of the designs $A(x)$ and $\hat{A}(x)$. Combining (2.7) and (2.8), we obtain

$$\int_0^l [\hat{A}(x)-A(x)][Eu'^2(x)-\rho\omega^2u^2(x)]\,dx \geq 0. \tag{2.9}$$

If the longitudinal displacement $u(x)$ of the design $A(x)$ satisfies

$$u'^2(x)-(\omega^2/c^2)u^2(x) = D^2, \tag{2.10}$$

where $c = \sqrt{(E/\rho)}$ is the speed of sound and D is a dimensionless constant, it follows from (2.9) that the design $A(x)$ cannot be heavier than any other design $\hat{A}(x)$ with the same dynamic response. Hence (2.10) is a sufficient condition for minimum-weight design. The proof that this is also a necessary condition for optimality will be postponed to the section on rods with segmentwise constant cross section because the proof given there includes continuously varying cross section as a special case.

The optimality condition (2.10) together with the kinematic boundary condition $u(0) = 0$ furnishes

$$u(x) = (c/\omega)D\sinh(\omega x/c). \tag{2.11}$$

The constant D is found by recalling that the value of $u(l)$ is given by the design constraint. Hence

$$D = \frac{(\omega/c)u(l)}{\sinh(\omega l/c)}. \tag{2.12}$$

Substitution of (2.11) into (2.1) yields

$$[\cosh(\omega x/c)]A'(x)+2(\omega/c)[\sinh(\omega x/c)]A(x) = 0. \tag{2.13}$$

Solving (2.13) for $A(x)$ we obtain

$$A(x) = \frac{\text{constant}}{\cosh^2(\omega x/c)}. \tag{2.14}$$

The integration constant in (2.14) is determined from the second equation of (2.2), and the minimum-weight design is found to be given by

$$A(x) = \frac{P(c/\omega)\cosh(\omega l/c)\sinh(\omega l/c)}{Eu(l)\cosh^2(\omega x/c)}. \tag{2.15}$$

The volume of material required for this optimal design is

$$V_0 = \int_0^l A(x)\,dx = [P/Eu(l)](c/\omega)^2\sinh^2(\omega l/c). \tag{2.16}$$

If V_p denotes the volume of the prismatic rod that has the same dynamic response, one readily finds that

$$\frac{V_0}{V_p} = \frac{\sinh^2(\omega l/c)}{(\omega l/c)\tan(\omega l/c)}. \tag{2.17}$$

This ratio as a function of $\omega l/c$ is shown in Fig. 2.

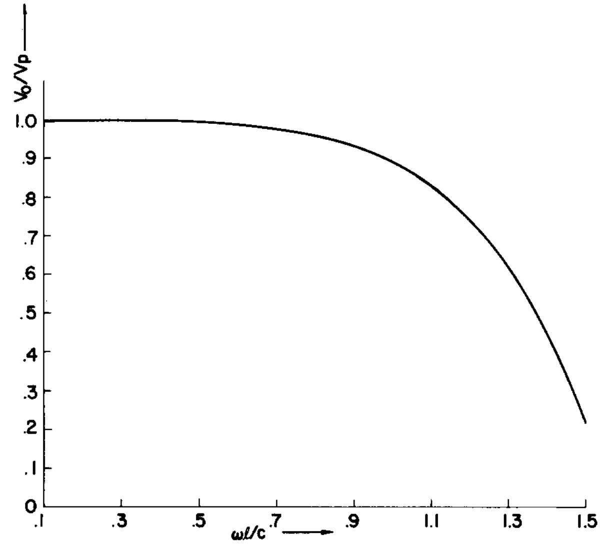


FIG. 2. Volume ratio vs. length ratio.

3. RODS WITH SEGMENTWISE CONSTANT CROSS SECTION

Consider a rod consisting of segments with constant cross section. Let the axial stiffness of the ith segment be EA_i. The weight of the ith segment is proportional to its axial stiffness. Thus, to minimize the weight of the rod is to minimize

$$W = \sum_i EA_i l_i, \tag{3.1}$$

where l_i is the length of the ith segment and the summation includes all segments. As before, the rod is supposed to be fixed at the end $x = 0$ and subjected to an axial load $P \cos \omega t$ at the end $x = l$. The axial displacement is written as $u(x) \cos \omega t$, and the dynamic response can be expressed as the minimum of the functional

$$\sum_i \left\{ \int_0^{l_i} EA_i \bar{u}_i'^2(x_i)\, \mathrm{d}x_i - \omega^2 \int_0^{l_i} \rho A_i \bar{u}_i^2(x_i)\, \mathrm{d}x_i \right\} \tag{3.2}$$

where x_i specifies a typical cross section of the ith span, and the functions $\bar{u}_i(x_i)$ for all segments represent a kinematically admissible field of axial displacements. The integrals in (3.2) are taken over the ith span and the summation includes all spans. If several segments are required to have the same axial stiffness, the term "span" refers to the sum of the lengths of all segments with the same axial stiffness.

Now consider two designs A_i and $\hat{A}_i$ that have the same dynamic response. It then follows from (3.2) that

$$\sum_i \left\{ \int_0^{l_i} EA_i u_i'^2(x_i)\, \mathrm{d}x_i - \omega^2 \int_0^{l_i} \rho A_i u_i^2(x_i)\, \mathrm{d}x_i \right\} = \sum_i \left\{ \int_0^{l_i} E\hat{A}_i \hat{u}_i'^2(x_i)\mathrm{d}x_i - \omega^2 \int_0^{l_i} \rho \hat{A}_i \hat{u}_i^2(x_i)\, \mathrm{d}x_i \right\}, \tag{3.3}$$

where $u_i(x_i)$ and $\hat{u}_i(x_i)$ are the longitudinal displacement fields of the designs A_i and $\hat{A}_i$, respectively. Combining (3.3) and (2.7) we obtain

$$\sum_i [\hat{A}_i - A_i] l_i \eta_i \geq 0, \tag{3.4}$$

where

$$\eta_i = \frac{1}{l_i} \int_0^{l_i} [u_i'^2(x_i) - (\omega^2/c^2) u_i^2(x_i)] \, dx_i. \tag{3.5}$$

Note that the two terms in the bracket of (3.5) are respectively proportional to the amplitudes of densities of strain energy and kinetic energy. From (3.4), (3.5), and (3.1) it follows that the design A_i cannot be heavier than any other design $\hat{A}_i$ with the same dynamic response if

$$\eta_1 = \eta_2 = \ldots = \eta_n, \tag{3.6}$$

where n is the number of spans making up the rod. Hence (3.6) is a sufficient condition of optimality.

In order to show that (3.6) is also necessary for optimality, consider first only two spans for simplicity. If ξ_i is defined as

$$\xi_i = [\hat{A}_i - A_i] l_i, \qquad (i = 1, 2) \tag{3.7}$$

the condition that the design $\hat{A}_i$ cannot be lighter than the minimum-weight design A_i becomes

$$\sum_i \xi_i \geq 0 \qquad (i = 1, 2). \tag{3.8}$$

On the other hand, it follows from (3.4) that

$$\sum_i \xi_i \eta_i \geq 0 \qquad (i = 1, 2). \tag{3.9}$$

Now, ξ_1, ξ_2, and η_1, η_2 are interpreted as the components of vectors ξ and η with respect to a rectangular Cartesian coordinate system. The optimal design A_1, A_2, and η_1, η_2 is unknown but fixed. The other design $\hat{A}_1$, $\hat{A}_2$ is only subject to the constraint of having a prescribed dynamic response. Accordingly, there exists a design $\hat{A}_1$, $\hat{A}_2$ giving the vector ξ an arbitrarily chosen direction pointing from the origin into the half plane defined by (3.8). The inequality (3.9) requires that the scalar product of any one of these vectors ξ with the unknown vector η be non-negative. The vector η must therefore be directed along the interior normal of the half plane (3.8) at the origin. This shows that the optimality condition (3.6) is necessary as well as sufficient. This proof of necessity is patterned on that given by Sheu and Prager [11]. Note that this proof of necessity is readily extended to the case of $n > 2$ spans of constant cross section. In fact, extending the proof to the case when $n \to \infty$ and all spans are infinitesimal, we readily establish the necessity of the optimality condition (2.10).

As an example of a rod with segmentwise constant cross section consider the rod in Fig. 3. The spans l_1 and l_2 are given and A_1, A_2 are to be determined to minimize the weight of the rod, which is to have given dynamic response. The differential equations of motion for the two segments are

$$u_i''(x_i)+k^2u_i(x_i) = 0 \qquad (i = 1, 2), \tag{3.10}$$

where $k^2 = \omega^2/c^2$.

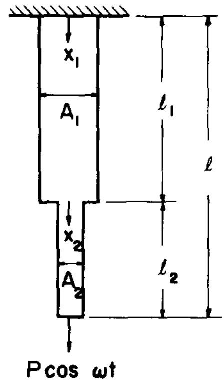


FIG. 3. Rod with segmentwise constant cross section.

With the boundary and transition conditions

$$\begin{aligned} &\text{at } x_1 = 0: && u_1(0) = 0, \\ &\text{at } x_1 = l_1: && u_1(l_1) = u_2(0), \\ &\phantom{\text{at }} x_2 = 0: && EA_1u_1'(l_1) = EA_2u_2'(0), \\ &\text{at } x_2 = l_2: && EA_2u_2'(l_2) = P, \end{aligned} \tag{3.11}$$

one finds the displacement fields

$$u_1(x_1) = \alpha \sin kx_1, \tag{3.12}$$

$$u_2(x_2) = \alpha[(A_1/A_2)\cos kl_1 \sin kx_2 + \sin kl_1 \cos kx_2], \tag{3.13}$$

where the constant α can be found by recalling that the dynamic tip displacement is to have the given value $u(l)$. Thus,

$$\alpha = \frac{u(l)}{[(A_1/A_2)\cos kl_1 \sin kl_2 + \sin kl_1 \cos kl_2]}. \tag{3.14}$$

The optimality condition (3.6) now becomes

$$\frac{1}{l_1}\int_0^{l_1} [u_1'^2(x_1)-k^2u_1^2(x_1)]\,\mathrm{d}x_1 = \frac{1}{l_2}\int_0^{l_2} [u_2'^2(x_2)-k^2u_2^2(x_2)]\,\mathrm{d}x_2. \tag{3.15}$$

Using integration by parts on the terms $\int_0^{l_1} u_1'^2(x_1)\,\mathrm{d}x_1$ and $\int_0^{l_2} u_2'^2(x_2)\,\mathrm{d}x_2$ in addition to the differential equations of motion (3.10), we find that the optimality condition (3.15) may be written in the alternative form

$$u_1(l_1)u_1'(l_1) = u_2(l_2)u_2'(l_2) - u_2(0)u_2'(0). \tag{3.16}$$

Substitution of (3.12) and (3.13) into (3.16) and use of appropriate trigonometric identities yields the following quadratic equation for the optimal value of A_1/A_2:

$$(A_1/A_2)^2[(\cos 2kl_1+1)\sin 2kl_2]+(A_1/A_2)[(\cos 2kl_2-1)\sin 2kl_1] \\ +(\cos 2kl_1-1)\sin 2kl_2-2\sin 2kl_1 = 0. \tag{3.17}$$

After obtaining the optimal value of A_1/A_2, from (3.17), we determine A_2 by using the fourth equation (3.11):

$$A_2 = \frac{P[(A_1/A_2)\cos kl_1 \sin kl_2+\sin kl_1 \cos kl_2]}{u(l)E(\omega/c)[(A_1/A_2)\cos kl_1 \cos kl_2-\sin kl_1 \sin kl_2]}. \tag{3.18}$$

The volume of material that is used in the rod is given by

$$V = A_1 l_1 + A_2 l_2. \tag{3.19}$$

In Fig. 4, the dimensionless structural volume

$$V^* = VEu(l)/(Pl^2) \tag{3.20}$$

is plotted vs. l_1/l for the various values of $\omega l/c$. When the value of l_1 is at the choice of the designer instead of being prescribed, it should be chosen to correspond to the abscissa of the lowest point of the curve for the given value of $\omega l/c$. Note that, in the range $0{\cdot}6 \leq \omega l/c \leq 1{\cdot}4$, the optimum value of l_1/l varies very little from about 0·500 to about 0·575.

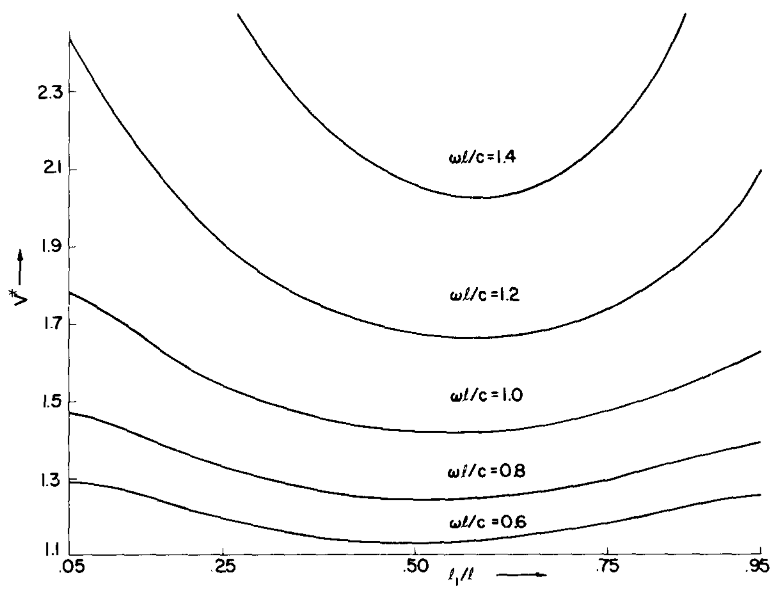


FIG. 4. Dimensionless volume vs. $\omega l/c$.

4. BEAM WITH CONTINUOUSLY VARYING BENDING STIFFNESS

Figure 5 shows a beam with continuously varying bending stiffness that is fixed at one end ($x = 0$) and subjected to the transverse load $P\cos \omega t$ at the other end ($x = l$). Both

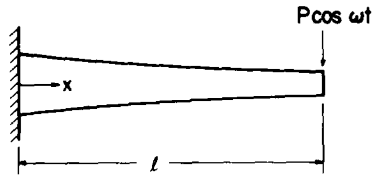


FIG. 5. Beam with continuously varying bending stiffness.

the amplitude P and the frequency ω are given, and *ω is to be smaller than the fundamental natural frequency ω_1* of the beam. Denoting the transverse deflection by $w(x)\cos\omega t$, we define the dynamic response of the beam to the given load as $Pw(l)$. The extremum principle characterizing the design constraint, which prescribes the value of $Pw(l)$, will be established using the method of Section 2.

The differential equation of motion is [10]

$$[EI(x)w''(x)]'' - \omega^2 m(x)w(x) = 0, \tag{4.1}$$

where E is Young's modulus, $m(x)$ is the mass distribution, $I(x)$ is the moment of inertia, and a prime denotes differentiation with respect to x. The boundary conditions are

$$\begin{aligned} &\text{at} \quad x = 0: \qquad w(0) = 0, \qquad w'(0) = 0, \\ &\text{at} \quad x = l: \qquad EI(l)w''(l) = 0, \qquad [EI(x)w''(x)]'_{x=l} = P. \end{aligned} \tag{4.2}$$

A continuous function $\bar{w}(x)$ in this problem is called kinematically admissible if it is twice differentiable with respect to x and satisfies the boundary conditions $\bar{w}(0) = \bar{w}'(0) = 0$; it need not, however, fulfill any boundary conditions at $x = l$.

Since ω_1 is assumed to exceed ω, it follows from Rayleigh's principle that the functional

$$F[\bar{w}(x)] = \int_0^l EI(x)\bar{w}''^2(x)\,dx - \omega^2 \int_0^l m(x)\bar{w}^2(x)\,dx \tag{4.3}$$

is positive definite. Proceeding essentially in the same manner as in Section 3, one readily establishes the following minimum principle:

$$\begin{aligned} &\int_0^l EI(x)\bar{w}''^2(x)\,dx - \omega^2 \int_0^l m(x)\bar{w}^2(x)\,dx - 2P\bar{w}(l) \\ &\quad \geq \int_0^l EI(x)w''^2(x)\,dx - \omega^2 \int_0^l m(x)w^2(x)\,dx - 2Pw(l) = -Pw(l). \end{aligned} \tag{4.4}$$

Note that this minimum principle reduces to the well-known principle of minimum potential energy if $\omega = 0$, which corresponds to static loading. Note also that the dynamic response $Pw(l)$ is the minimum of the functional $F[\bar{w}(x)]$.

Now, consider two designs $I(x)$ and $\hat{I}(x)$ with the same dynamic response. Accordingly,

$$\begin{aligned} &\int_0^l EI(x)w''^2(x)\,dx - \omega^2 \int_0^l m(x)w^2(x)\,dx \\ &\quad = \int_0^l E\hat{I}(x)\hat{w}''^2(x)\,dx - \omega^2 \int_0^l \hat{m}(x)\hat{w}^2(x)\,dx, \end{aligned} \tag{4.5}$$

where $w(x)$ and $\hat{w}(x)$ are the deflections of the designs $I(x)$ and $\hat{I}(x)$. The mass–stiffness relation for sandwich beams with constant height of core has the form

$$m(x) = a^2 + b^2 EI(x), \tag{4.6}$$

where a and b are constants. Combining (4.4), (4.5), and (4.6), we obtain

$$\int_0^l E[\hat{I}(x) - I(x)][w''^2(x) - \omega^2 b^2 w^2(x)]\,dx \geq 0. \tag{4.7}$$

If the transverse deflection $w(x)$ of the design $I(x)$ satisfies

$$w''^2(x) - \omega^2 b^2 w^2(x) = H^2, \tag{4.8}$$

where H is a constant, it follows from (4.7) that the design $I(x)$ cannot be heavier than any other design $\hat{I}(x)$ with the same dynamic response. Hence (4.8) is a sufficient condition for minimum-weight design. The proof that this is also a necessary condition for optimality follows the same lines as the necessity proof in Section 3.

In order to obtain the optimal design, the deflection of the optimal structure must be found from (4.8). However the optimality condition (4.8) is a nonlinear differential equation, which cannot be integrated in closed form. A numerical solution $w_1(x)$ must therefore be obtained for an arbitrary value of the constant H, say $H = 1$, under the boundary conditions in the first line of (4.2). Note that for any other value of H, we have

$$w(x) = Hw_1(x). \tag{4.9}$$

Since $w_1''(l)$ is not likely to vanish, it follows from the first equation in the second line of (4.2) that

$$I(l) = 0. \tag{4.10}$$

Substituting (4.6) and (4.9) into (4.1), and using (4.8), one obtains the following differential equation for $I(x)$:

$$[EI(x)\sqrt{(1 + \omega^2 b^2 w_1^2(x))}]'' - \omega^2[a^2 + b^2 EI(x)]w_1(x) = 0. \tag{4.11}$$

This must be integrated under the boundary conditions (4.2), (4.10), and

$$I'(l) = P/[EHw_1''(l)]. \tag{4.12}$$

The last condition follows from the second equation in the second line of (4.2), (4.9), and (4.10). The constant H can be found from the design constraint of a given dynamic response $Pw(l)$. Since P is given, we know $w(l)$, and H is then obtained from (4.9).

5. OPTIMAL DESIGN OF TRUSSES

Consider a generalized truss that is loaded at the typical joint i by $\underline{P}_i \cos \omega t$ and carries a mass M_i at this joint. All masses are assumed to be lumped at the joints. The truss is to be designed for minimum weight when the dynamic response $\sum_i \underline{P}_i \cdot \underline{u}_i$ is prescribed, the center dot indicating the scalar product. In order to obtain the optimality condition, the following function will be used in the same manner as (2.4):

$$F[\underline{u}_i^*] = \tfrac{1}{2} \sum_i \sum_j \alpha_{ij} s_{ij} \lambda_{ij}^2 - \omega^2 \sum_i M_i \underline{u}_i^* \cdot \underline{u}_i^*, \tag{5.1}$$

where $\underline{u}_i^* \cos \omega t$ is a kinematically admissible displacement vector of the joint i, and s_{ij} is the specific axial stiffness of the bar connecting joints i and j (i.e. EA_{ij}/l_{ij} where A_{ij} and l_{ij} are the cross-sectional area and the length of the bar connecting joints i and j). The elongation amplitudes λ_{ij} are defined by

$$\lambda_{ij} = (\underline{u}_i^* - \underline{u}_j^*) \,.\, \underline{e}_{ij}, \tag{5.2}$$

where $\underline{e}_{ij}$ is a unit vector along the ray from joint i to joint j. The quantities α_{ij} are defined by

$$\alpha_{ij} = 0 \quad \text{if } i \text{ and } j \text{ are not connected by a bar,}$$

$$\alpha_{ij} = 1 \quad \text{if } i \text{ and } j \text{ are connected by a bar.}$$

Displacements $\underline{u}_i^*$ are kinematically admissible if they satisfy the kinematic constraints imposed by the supports of the truss. Note that the factor $\frac{1}{2}$ in the first term on the right-hand side of (5.1) is needed, because the double sum involves each bar twice.

For the actual displacement amplitudes $\underline{u}_i$, $\underline{u}_j$ and any kinematically admissible displacement amplitudes $\underline{u}_i^*$, $\underline{u}_j^*$, we have

$$\tfrac{1}{2}\sum_i \sum_j \alpha_{ij} s_{ij} \{[\underline{u}_i - \underline{u}_j - (\underline{u}_i^* - \underline{u}_j^*)] \,.\, \underline{e}_{ij}\}^2 - \omega^2 \sum_i M_i (\underline{u}_i - \underline{u}_i^*) \,.\, (\underline{u}_i - \underline{u}_i^*) \geq 0. \tag{5.3}$$

Expansion of the squared term and subtraction of

$$\sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i - \underline{u}_j) \,.\, \underline{e}_{ij}]^2 - 2\omega^2 \sum_i M_i \underline{u}_i \,.\, \underline{u}_i$$

from both sides of the resulting inequality yields

$$\begin{aligned}
&\tfrac{1}{2}\sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i^* - \underline{u}_j^*) \,.\, \underline{e}_{ij}]^2 - \omega^2 \sum_i M_i \underline{u}_i^* \,.\, \underline{u}_i^* \\
&\quad - \sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i - \underline{u}_j) \,.\, \underline{e}_{ij}]\,[(\underline{u}_i^* - \underline{u}_j^*) \,.\, \underline{e}_{ij}] + 2\omega^2 \sum_i M_i \underline{u}_i \,.\, \underline{u}_i^* \\
&\quad \geq \tfrac{1}{2}\sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i - \underline{u}_j) \,.\, \underline{e}_{ij}]^2 - \omega^2 \sum_i M_i \underline{u}_i \,.\, \underline{u}_i \\
&\quad - \sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i - \underline{u}_j) \,.\, \underline{e}_{ij}]^2 + 2\omega^2 \sum_i M_i \underline{u}_i \,.\, \underline{u}_i.
\end{aligned} \tag{5.4}$$

The principle of virtual work for this truss states that

$$\tfrac{1}{2}\sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i - \underline{u}_j) \,.\, \underline{e}_{ij}]^2 = \omega^2 \sum_i M_i \underline{u}_i \,.\, \underline{u}_i + \sum_i \underline{P}_i \,.\, \underline{u}_i, \tag{5.5}$$

and

$$\tfrac{1}{2}\sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i - \underline{u}_j) \,.\, \underline{e}_{ij}]\,[(\underline{u}_i^* - \underline{u}_j^*) \,.\, \underline{e}_{ij}] = \omega^2 \sum_i M_i \underline{u}_i \,.\, \underline{u}_i^* + \sum_i \underline{P}_i \,.\, \underline{u}_i^*, \tag{5.6}$$

where $\underline{P}_i$ is the amplitude of the load acting at the joint i. Substitution of (5.5) and (5.6) into (5.4) yields the following minimum principle:

$$\begin{aligned}
&\tfrac{1}{2}\sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i^* - \underline{u}_j^*) \,.\, \underline{e}_{ij}]^2 - \omega^2 \sum_i M_i \underline{u}_i^* \,.\, \underline{u}_i^* - 2\sum_i \underline{P}_i \,.\, \underline{u}_i^* \\
&\quad \geq \tfrac{1}{2}\sum_i \sum_j \alpha_{ij} s_{ij} [(\underline{u}_i - \underline{u}_j) \,.\, \underline{e}_{ij}]^2 - \omega^2 \sum_i M_i \underline{u}_i \,.\, \underline{u}_i - 2\sum_i \underline{P}_i \,.\, \underline{u}_i.
\end{aligned} \tag{5.7}$$

Consider now the design s_{ij} and an alternative design $\hat{s}_{ij}$ with the same dynamic response. Since the minimum value of the functional (5.1) must be the same for the two designs s_{ij} and $\hat{s}_{ij}$:

$$\tfrac{1}{2}\sum_i\sum_j \alpha_{ij}s_{ij}[(\underline{u}_i-\underline{u}_j)\cdot\underline{e}_{ij}]^2-\omega^2\sum_i M_i\underline{u}_i\cdot\underline{u}_i = \tfrac{1}{2}\sum_i\sum_j \alpha_{ij}\hat{s}_{ij}[(\hat{\underline{u}}_i-\hat{\underline{u}}_j)\cdot\underline{e}_{ij}]^2-\omega^2\sum_i M_i\hat{\underline{u}}_i\cdot\hat{\underline{u}}_i, \tag{5.8}$$

where u_i, u_j and $\hat{u}_i$, $\hat{u}_j$ are the displacement amplitudes of the designs s_{ij} and $\hat{s}_{ij}$. Combining (5.7) and (5.8) we obtain

$$\tfrac{1}{2}\sum_i\sum_j \alpha_{ij}[\hat{s}_{ij}-s_{ij}][(\underline{u}_i-\underline{u}_j)\cdot\underline{e}_{ij}]^2 \geq 0. \tag{5.9}$$

Note that the weight of the truss is proportional to $\sum_i\sum_j A_{ij}l_{ij}$ which is proportional to $\sum_i\sum_j s_{ij}l_{ij}^2$ since $s_{ij} = EA_{ij}/l_{ij}$. If the displacements $\underline{u}_i$ of the design s_{ij} satisfy

$$[(\underline{u}_i-\underline{u}_j)\cdot\underline{e}_{ij}]^2 = \varepsilon^2 l_{ij}^2, \tag{5.10}$$

where ε is a dimensionless constant representing the absolute value of the unit extension of the bar connecting the joints i and j, it then follows from (5.10) that the design s_{ij} cannot be heavier than any other design $\hat{s}_{ij}$ with the same dynamic response. Hence (5.10) is a sufficient condition for minimum-weight design of trusses. The proof that this is also a necessary condition for optimality uses the same kind of argument as the necessity proof in Section 3. Use of the optimality condition (5.10) is illustrated by the following two examples.

Consider the truss in Fig. 6 supporting equal concentrated masses M at the joints 4, 6, and 8 subjected to the load $P\cos\omega t$ applied at joint 8. The masses of the bars are assumed to be negligible in comparison to the lumped masses M. The optimality condition (5.10) requires that all bars experience a unit extension of the same absolute value ε. The convention of considering tension as positive and compression as negative will be adopted for the bar forces. The expected signs of the bar forces are indicated in Fig. 6.

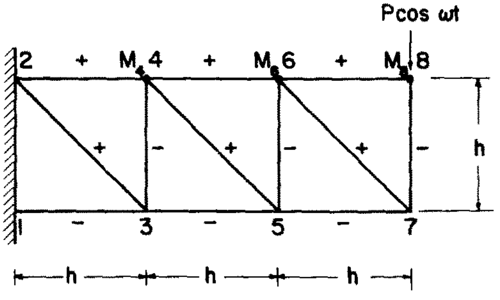


FIG. 6. Truss A.

From the Williot diagram in Fig. 7, the joint displacements in multiples of εh are found to be the following:

	Components	
Joint	Rightward horizontal	Downward vertical
1	0	0
2	0	0
3	−1	3
4	1	4
5	−2	9
6	2	10
7	−3	17
8	3	18

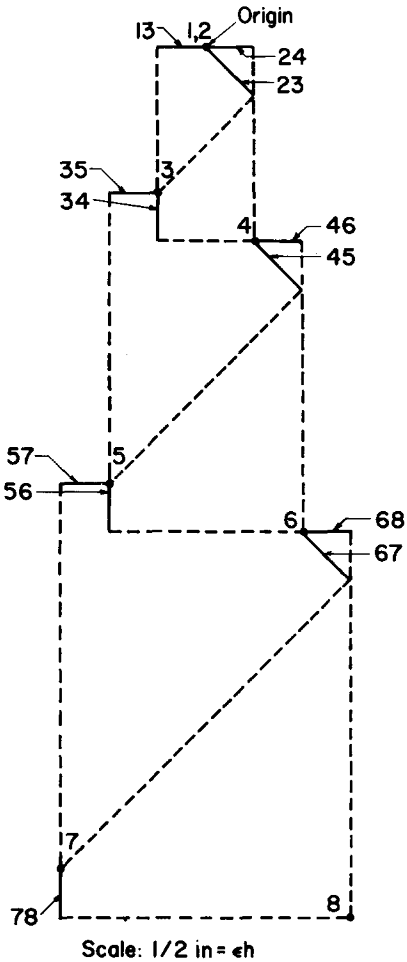
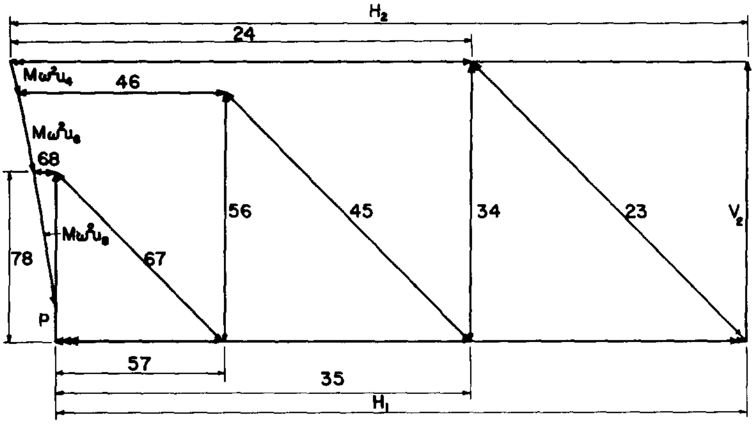


FIG. 7. Williot diagram for truss A.

Assuming the magnitude of the load $P\cos\omega t$ to be $4M\omega^2\varepsilon h$, the bar forces and support reactions are found by using the Maxwell diagram in Fig. 8. The forces and reactions in multiples of $M\omega^2\varepsilon h$ are:

Bar	Axial force
78	−22
68	3
57	−22
67	31·1
56	−32
46	27
35	−54
45	45·3
34	−36
24	60
23	50·9
13	−90
Reaction	**Magnitude**
Vertical at 1	0
Horizontal at 1	90 (rightward)
Vertical at 2	36 (upward)
Horizontal at 2	96 (leftward)



Scale: 1/2 in = $5M\omega^2\varepsilon h$

FIG. 8. Maxwell diagram for truss A.

The signs of these bar forces must be compared to those assumed for the Williot diagram in Fig. 7, from which the inertia forces were derived. If there should be a contradiction, the Williot diagram must be reconstructed with the appropriate changes in sign. No change was required here.

The truss in Fig. 9 supports equal concentrated masses M at the joints 4, 7, and 9, and is subjected to the load $P \cos \omega t$ applied at joint 9. Again the masses of the bars are assumed to be negligible in comparison to the lumped masses M. The optimality condition (5.10) requires that all bars experience a unit extension of the same absolute value ε.

After the signs of the bar forces have been assumed as shown in Fig. 9, the Williot diagram in Fig. 10 is constructed. The joint displacements in multiples of εh are found to be the following:

Joint	Components	
	Rightward horizontal	Downward vertical
1	0	0
2	0	0
3	−1	3
4	0	2
5	1	3
6	−2	7
7	0	6
8	2	7
9	0	11

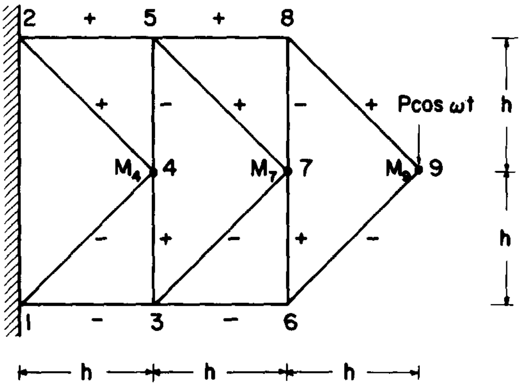


FIG. 9. Truss *B*.

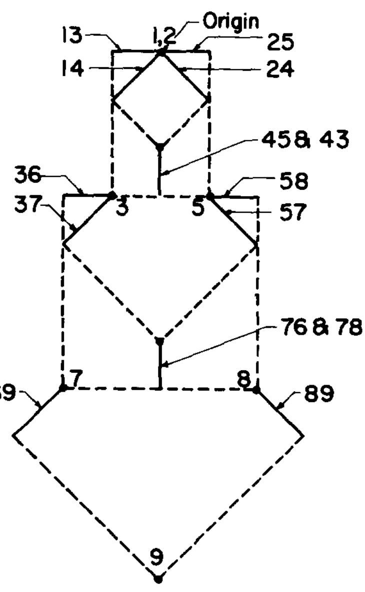


FIG. 10. Williot diagram for truss *B*.

Again, assuming the magnitude of the load $P \cos \omega t$ to be $4M\omega^2 \varepsilon h$, the bar forces and support reactions are found by using the Maxwell diagram in Fig. 11. The forces and reactions in multiples of $M\omega^2 \varepsilon h$ are:

Bar		Axial force
69		−10·6
89		10·6
78		−7·5
58		7·5
36		−7·5
67		7·5
37		−15·9
57		15·9
45		−10·5
25		18·0
13		−18·0
34		10·5
14		−16·3
24		16·3
Reaction		Magnitude
Vertical	at 1	11·5 (upward)
Horizontal	at 1	29·5 (rightward)
Vertical	at 2	11·5 (upward)
Horizontal	at 2	29·5 (leftward)

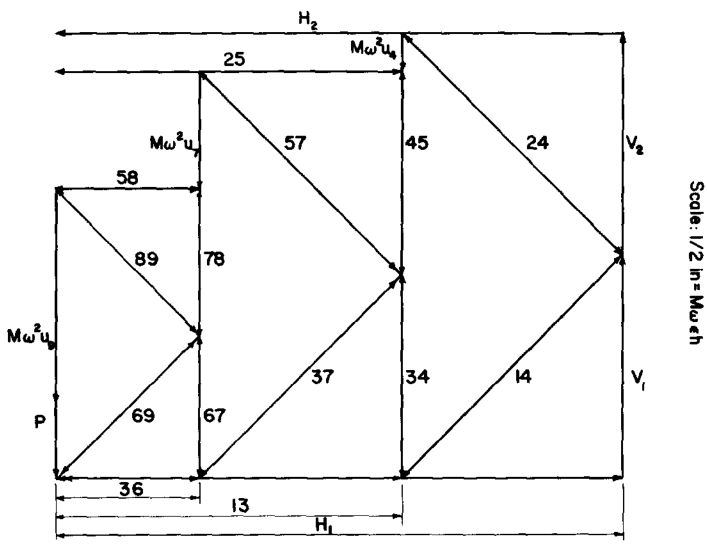


FIG. 11. Maxwell diagram for truss B.

Again, the signs of the bar forces assumed for the Williot diagram proved to be correct.

The principle of virtual work can be used to calculate the total volume of material used in the truss. The principle of virtual work for a truss states that

$$W = \tfrac{1}{2}\sum_i \sum_j F_{ij}\lambda_{ij} = \sum_i (\underline{P}_i + \omega^2 M_i \underline{u}_i) \cdot \underline{u}_i, \tag{5.11}$$

where W is the work done by the applied forces $\underline{P}_i \cos \omega t$ and the inertia forces $M\omega^2 \underline{u}_i$, and F_{ij} is the magnitude of the axial force in the bar connecting joints i and j. We also know that

$$F_{ij} = E\varepsilon A_{ij}, \tag{5.12}$$

and

$$\lambda_{ij} = \varepsilon l_{ij}. \tag{5.13}$$

From (5.11), (5.12), and (5.13) we obtain

$$V = W/(E\varepsilon^2) = \sum_i (\underline{P}_i + \omega^2 M_i \underline{u}_i) \,.\, \underline{u}_i/(E\varepsilon^2) \tag{5.14}$$

where V is the volume of material used in the truss. The weight of the truss is proportional to V. Hence the weight of each of the two considered trusses can be calculated.

The total weight of truss A is

$$T = 526(M/E)\omega^2 h^2 \rho, \tag{5.15}$$

where ρ is the density of the material from which the truss is constructed and T is the weight of the truss. The total weight of truss B is

$$T = 205(M/E)\omega^2 h^2 \rho. \tag{5.16}$$

Note the considerable saving of weight possible by the different layout of the bars in the truss. When the layout is not prescribed, the minimum-weight is obtained with a Michell truss [12].

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REFERENCES

[1] R. L. Barnett, Minimum weight design of beams for deflection. *J. Engng Mech. Div. Am. Soc. civ. Engrs* **128**, 75–109 (1961).
[2] E. J. Haug and P. G. Kirmser, Minimum weight design of beams with inequality constraints on stress and deflection. *J. appl. Mech.* **34**, 999–1004 (1967).
[3] F. I. Niordsen, On the optimal design of a vibrating beam. *Q. appl. Math.* **33**, 47–53 (1965).
[4] J. M. Turner, Design of minimum-mass structures with specified natural frequencies. *AIAA Jnl* **5**, 406–412 (1967).
[5] J. E. Taylor, Minimum-mass bar for axial vibration at specified natural frequency. *AIAA Jnl* **5**, 1911–1913 (1967).
[6] M. S. Zarghamee, Optimum frequency of structures. *AIAA Jnl* **5**, 749–750 (1967).
[7] C. Y. Sheu, Elastic minimum-weight design for specified fundamental frequency. *Int. J. Solids Struct.* **4**, 953–958 (1968).
[8] W. Prager and J. E. Taylor, Problems of optimal structural design. *J. appl. Mech.* **35**, 102–106 (1968).
[9] W. Prager and R. T. Shield, Optimal design of multi-purpose structures. *Int. J. Solids Struct.* **4**, 469–475 (1968).
[10] W. Nowachi, *Dynamics of Elastic Systems*, pp. 83 and 113. Chapman & Hall (1963).
[11] C. Y. Sheu and W. Prager, Minimum-weight design with piecewise constant specific stiffness. *J. Optimization Theory Applic.* **2**, 179–186 (1968).
[12] A. G. M. Michell, The limits of economy of material in frame-structures. *Phil. Mag.* Ser. 6, **8**, 589–597 (1904).

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Абстракт—Работа занимается оптимальном проектированием конструкций разных типов, вследствие возбуждения к гармоническим вибрациям через одинарный груз, при интенсивности изменяющейся гармонически во времени. Амплитуда и частота этой нагрузки определяются как "динамическая реакция" конструкции, названна виртуальной работой амплитуды нагрузки на перемещении амплитуды в точке её применения. Предметом "принуждения проектирования" конструкции является использование наименьшего возможного количества конструкционного материала. Приводятся необходимые и достаточные условия оптимализации для стержней и балок, с поперечными сечениями, изменяющимися непреривно или кусочно постоянными отрезками, а также для ферм, где все массы расположены в точках связи. Даются примеры, иллюстрирующие возможное уменьшение веса конструкции.